# NONLINEAR DAMPING OF THE NATURAL VIBRATIONS OF SYSTEMS OF ARBITRARY ORDER

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Nonlinear damping of the natural vibrations of a system of order $2n$ will be considered.

Linear damping of the natural vibrations of the second order system

$$\ddot{x} + \omega^2 x = \delta \qquad (1)$$

is due to the fact that in the presence of the damping term $\delta = -2|h|\chi$ one is dealing with the linear equation with damped solutions

$$\ddot{x} + 2|h|\dot{x} + \omega^2 x = 0$$

The damping of natural vibrations of the system (1) by a nonlinear damping term $\delta$ (which does not depend on the quantity $\chi$) will be studied.

Let the system (1) with a nonlinear damping term be described by the equation

$$\ddot{x} + \omega^2 (1 - k^2) x = 0 \qquad (2)$$

where $k^2 = 0$ for $x\chi > 0$, $0 < k^2 < 1$ for $x\chi < 0$. Then, at the instant corresponding to $\chi = 0 (x = x_{max})$, the energy of the system (2) has decreased to the extreme value $1/2\, k^2 \omega^2 x^2_{max_n}$ and the solution $x$ of the equation (2) will be damped.

The time it takes to reduce $x$ to a value not exceeding $0.05\, x_{max}$ for the initial conditions $x(0) = 0$, $\chi(0) = \omega x_{max}$ is given by the formula

$$t \leqslant \frac{\pi}{2\omega}\left[\left(1 + \frac{1}{\sqrt[m]{0.05}}\right) m + 1\right]$$

In this formula, $m$ is the number of cycles leading to the damping and $x_{max}$ is the absolute value of the maximum of $x$. The described method of nonlinear damping will be applied to the damping of the natural oscillations of a system of order $2n$ without use of generalized velocities.

Consider the general case of the equations of the natural vibrations of a system with two degrees of freedom without friction

$$\beta_{11}\ddot{x}_1 + \beta_{12}\ddot{x}_2 + \alpha_{11}x_1 + \alpha_{12}x_2 = 0, \beta_{12}\ddot{x} + \beta_{22}\ddot{x}_2 + \alpha_{12}x_1 + \alpha_{22}x_2 = 0 \qquad (3)$$

where

$$\alpha_{11} > 0, \qquad \alpha_{22} > 0, \qquad \alpha_{11}\alpha_{22} > \alpha_{12}^2 \qquad (4)$$

$$\beta_{11} > 0, \qquad \beta_{22} > 0, \qquad \beta_{11}\beta_{22} > \beta_{12}^2 \qquad (5)$$

The inequalities (4) are the conditions for the system (3) to have positive potential energy, since the natural vibrations are a motion about the stable equilibrium position $x_1 = x_2 = 0$. The kinetic energy will always be positive as a result of the inequalities (5).

The solution of the system (3) has the form

$$x_1 = \xi + \eta, \qquad x_2 = k_1\xi + k_2\eta \qquad (6)$$

where

$$\xi = A_1 \cos(\omega_1 t + \varphi_1), \qquad \eta = A_2 \cos(\omega_2 t + \varphi_2)$$

Here $A_1$, $A_2$, $\phi_1$, $\phi_2$ are determined by the initial conditions and $\omega_1$, $\omega_2$ by the equation

$$\begin{vmatrix} \alpha_{11} - \omega^2\beta_{11} & \alpha_{12} - \omega^2\beta_{12} \\ \alpha_{12} - \omega^2\beta_{12} & \alpha_{22} - \omega^2\beta_{22} \end{vmatrix} = 0$$

the roots of which will be real on the strength of the inequalities (4), (5). The amplitude coefficients are given by

$$k_1 = -\frac{\alpha_{11} - \omega_1^2\beta_{11}}{\alpha_{12} - \omega_1^2\beta_{12}} = -\frac{\alpha_{12} - \omega_1^2\beta_{12}}{\alpha_{22} - \omega_1^2\beta_{22}}, \quad k_2 = -\frac{\alpha_{11} - \omega_2^2\beta_{11}}{\alpha_{12} - \omega_2^2\beta_{12}} = -\frac{\alpha_{12} - \omega_2^2\beta_{12}}{\alpha_{22} - \omega_2^2\beta_{22}} \qquad (7)$$

Let in the equations with the two damping terms $\delta_1$, $\delta_2$

$$\beta_{11}\ddot{x}_1 + \beta_{12}\ddot{x}_2 + \alpha_{11}x_1 + \alpha_{12}x_2 = \delta_1, \quad \beta_{12}\ddot{x}_1 + \beta_{22}\ddot{x}_2 + \alpha_{12}x_1 + \alpha_{22}x_2 = \delta_2 \qquad (8)$$

the displacements $x_1$, $x_2$ be related to the $\xi, \eta$ by the formulas (6), but let $\xi, \eta$ satisfy the equations

$$\ddot{\xi} + \omega_1^2(1 - k_\xi^2)\xi = 0, \qquad \ddot{\eta} + \omega_2^2(1 - k_\eta^2)\eta = 0 \qquad (9)$$

where

$$\begin{aligned} k_\xi^2 &= 0 \quad \text{for } \xi\dot{\xi} > 0, \qquad 0 < k_\xi^2 < 1 \quad \text{for } \xi\dot{\xi} < 0 \\ k_\eta^2 &= 0 \quad \text{for } \eta\dot{\eta} > 0, \qquad 0 < k_\eta^2 < 1 \quad \text{for } \eta\dot{\eta} < 0 \end{aligned} \qquad (10)$$

Then $\chi_1$, $\chi_2$ are stepwise continuous and given by

$$\begin{aligned} \ddot{x}_1 &= \ddot{\xi} + \ddot{\eta} = -\omega_1^2(1 - k_\xi^2)\xi - \omega_2^2(1 - k_\eta^2)\eta \\ \ddot{x}_2 &= k_1\ddot{\xi} + k_2\ddot{\eta} = -k_1\omega_1^2(1 - k_\xi^2)\xi - k_2\omega_2^2(1 - k_\eta^2)\eta \end{aligned} \qquad (11)$$

Substituting (6), (11) in the equations (8) and using (7), one finds

$$\delta_1 \equiv (\beta_{11} + k_1\beta_{12})\,\omega_1^2 k_\xi^2 \xi + (\beta_{11} + k_2\beta_{12})\,\omega_2^2 k_\eta^2 \eta$$
$$\delta_2 \equiv (\beta_{12} + k_1\beta_{22})\,\omega_1^2 k_\xi^2 \xi + (\beta_{12} + k_2\beta_{22})\,\omega_2^2 k_\eta^2 \eta$$

As a result, the solution of the equations (8) will be stable in the final region (10). Since it follows from (6) that

$$\xi = \frac{k_2 x_1 - x_2}{k_2 - k_1}, \qquad \eta = \frac{x_2 - k_1 x_1}{k_2 - k_1} \tag{12}$$

one has

$$\delta_1 \equiv (\beta_{11} + k_1\beta_{12})\,\omega_1^2 k_\xi^2 \frac{k_2 x_1 - x_2}{k_2 - k_1} + (\beta_{11} + k_2\beta_{12})\,\omega_2^2 k_\eta^2 \frac{x_2 - k_1 x_1}{k_2 - k_1}$$
$$\delta_2 \equiv (\beta_{12} + k_1\beta_{22})\,\omega_1^2 k_\xi^2 \frac{k_2 x_1 - x_2}{k_2 - k_1} + (\beta_{12} + k_2\beta_{22})\,\omega_2^2 k_\eta^2 \frac{x_2 - k_1 x_1}{k_2 - k_1}$$

where $k_\xi^2$, $k_\eta^2$ depend on the signs of $(k_2x_1 - x_2)d/dt(k_2x_1 - x_2)$, $(x_2 - k_1x_1)d/dt(x_2 - k_1x_1)$ respectively. Thus, $\delta_1$, $\delta_2$ are sectionally continuous functions of $x_1$, $x_2$ which do not depend on the magnitude of the derivatives $\chi_1$, $\chi_2$.

Similarly, one may solve the general problem of the damping of the natural vibrations of a system with $n$ degrees of freedom without friction

$$\sum_{s=1}^{n} (\beta_{sl}\ddot{x}_s + \alpha_{sl}x_s) = \delta_l \qquad (l = 1, \ldots, n) \tag{13}$$

by the help of $n$ damping terms $\delta_l$ which do not depend on the magnitudes of the derivatives $\chi_1$, ..., $\chi_n$. Here $\alpha_{sl}$, $\beta_{sl}$ are such that the solution of the equations (13) for $\delta_1 = 0$ takes the form

$$x_s = \sum_{p=1}^{n} A_{1p} k_{sp} \cos(\omega_p t + \varphi_p) \qquad (s = 1, \ldots, n)$$

The constants $A_{11}$, ..., $A_{1n}$, $\phi_1$, ..., $\phi_n$ are determined by the initial conditions, $\omega_1$, ..., $\omega_u$ are the eigenvalues of the system (13) for $\delta_l = 0$, the $k_{sp}^{*}$ are determined by the system of equations

$$\sum_{s=2}^{n} (\alpha_{sl} - \omega_p^2\beta_{sl})\,k_{sp} = \beta_{1l}\omega_p^2 - \alpha_{1l} \qquad (l = 1, \ldots, n-1)$$

The presence of friction in the system does not disturb the effect of the nonlinear damping term, whose equations are found by studying the conservative systems (3), (13). If one adds to the systems (8), (13) negative friction, sufficiently small so that the solutions of the corresponding equations without damping terms are vibrationally unstable, then by the strength of the extent of the region of stability (10), the solutions of the equations (8), (13) with damping terms may provide for the predominance of the damping effect of the nonlinear damping terms over the oscillations. In other words, it will be required that in the interval between two zeros of the function of the time $\xi$ (or $\eta$) the expenditure of energy of the system (8) at the instant ( $\xi = 0$ or $\eta = 0$ ), including the damping terms $\delta_1$, $\delta_2$, exceeds the supply of energy due to the negative friction.

In the equations (8), let $\delta_1 = 0$, whence $x_1$ satisfies an equation of the form

$$x_1^{(4)} + 2a\ddot{x}_1 + bx_1 = c\delta_2 + d\ddot{\delta}_2 \qquad (a > 0,\ b > 0,\ a^2 > b) \tag{14}$$

One obtains then from equations (9), (11) the stepwise continuous function of time

$$x_1^{(4)} = \xi^{(4)} + \eta^{(4)} = \omega_1^4 (1 - k_\xi^2)^2 \xi + \omega_2^4 (1 - k_\eta^2)^2 \eta \tag{15}$$

By (6), (11), (15), and since $\omega^4 - 2a\omega^2 + b = 0$, one has

$$c\delta_2 + d\ddot{\delta}_2 = f_1(k_\xi^2)\xi + f_2(k_\eta^2)\eta \qquad \begin{pmatrix} f_1(k_\xi^2) = \omega_1^4 (k_\xi^4 - 4k_\xi^2) - bk_\xi^2 \\ f_2(k_\eta^2) = \omega_2^4 (k_\eta^4 - 4k_\eta^2) - bk_\eta^2 \end{pmatrix}$$

Let $\delta_2 = n_1(k_\xi^2)\xi + n_2(k_\eta^2)\eta$, where $n_1(k_\xi^2)$, $n_2(k_\eta^2)$ are certain combinations of the numbers $k_\xi^2$, $k_\eta^2$, then

$$\ddot{\delta}_2 = -n_1(k_\xi^2)\omega_1^2(1 - k_\xi^2)\xi - n_2(k_\eta^2)\omega_2^2(1 - k_\eta^2)\eta$$

$$n_1(k_\xi^2) = \frac{f_1(k_\xi^2)}{-d\omega_1^2(1 - k_\xi^2) + c}, \qquad n_2(k_\eta^2) = \frac{f_2(k_\eta^2)}{-d\omega_2^2(1 - k_\eta^2) + c}$$

It follows from the equations (6), (11) that $\xi, \eta$ may be expressed in terms of $x_1$ and $\chi_1$

$$\xi = \frac{\ddot{x}_1 + \omega_2^2(1 - k_\eta^2)x_1}{\omega_2^2(1 - k_\eta^2) - \omega_1^2(1 - k_\xi^2)}, \qquad \eta = \frac{\ddot{x}_1 + \omega_1^2(1 - k_\xi^2)x_1}{\omega_1^2(1 - k_\eta^2) - \omega_2^2(1 + k_\xi^2)} \tag{16}$$

Here $k_\eta^2$ depend on

$$\operatorname{sign}[\ddot{x}_1 + \omega_2^2(1 - k_\eta^2)x_1]\frac{d}{dt}[\ddot{x}_1 + \omega_2^2(1 - k_\eta^2)x_1]$$

$$\operatorname{sign}[\ddot{x}_1 + \omega_1^2(1 - k_\xi^2)x_1]\frac{d}{dt}[\ddot{x}_1 + \omega_1^2(1 - k_\xi^2)x_1]$$

respectively. Consequently, the equation for the nonlinear damping term may be written

$$\delta_2 = n_1(k_\xi^2)\frac{\ddot{x}_1 + \omega_2^2(1 - k_\eta^2)x_1}{\omega_2^2(1 - k_\eta^2) - \omega_1^2(1 - k_\xi^2)} + n_2(k_\eta^2)\frac{\ddot{x}_1 + \omega_1^2(1 - k_\xi^2)x_1}{\omega_1^2(1 - k_\eta^2) - \omega_2^2(1 - k_\xi^2)}$$

when the linear damping of the natural vibrations of the system (14) depends likewise on the magnitude of the derivatives $\chi_1$ and $\chi_1$.

Instead of the formulas (16) for $\xi, \eta$, one may use the equivalents (12), when

$$\delta_2 = n_1(k_\xi^2)\frac{k_2x_1 - x_2}{k_2 - k_1} + n_2(k_\eta^2)\frac{x_2 - k_1x_1}{k_2 - k_1}$$

where $k_\xi^2$, $k_\eta^2$ depend on the signs of the expressions

$$(k_2x_1 - x_2)d/dt\,(k_2x_1 - x_2),\quad (x_2 - k_1x_1)d/dt\,(x_2 - k_1x_1)$$

respectively.

Similarly, one may solve the problem of nonlinear damping of natural vibrations of the system (13) with one damping term which does not depend on the magnitude of $\chi_i$ ($i = 1, \ldots, n$).

*Translated by* J.R.M.R.